On Stationary Solutions of Delay Differential Equations Driven by a Lévy Process*

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November 24, 1998

Abstract

The stochastic delay differential equation

$$dX(t) = \int\limits_{[-r,0]} X(t+u)\,a(du) + dZ(t), \qquad t \geq 0,$$

is considered, where $Z(t)$ is a process with independent stationary increments and a is a finite signed measure. We obtain necessary and sufficient conditions for the existence of a stationary solution to this equation in terms of a and the Lévy measure of Z.

Keywords: Lévy processes; processes of Ornstein–Uhlenbeck type; stationary solution; stochastic delay differential equations

1 Introduction

Let a be a finite signed measure on a finite interval $J = [-r, 0]$, $r \geq 0$. Consider the equation

$$\begin{aligned} X(t) &= X(0) + \int\limits_0^t \int\limits_J X(s+u)\,a(du)\,ds + Z(t), && t \geq 0, \\ X(t) &= X_0(t), && t \in J. \end{aligned} \tag{1.1}$$

Here $Z = (Z(t),\, t \geq 0)$ is a real-valued process with independent stationary increments starting from 0 and having càdlàg trajectories, i.e. Z is a Lévy process, and $X_0 =$

* The research on this paper was carried out within the Sonderforschungsbereich 373 at Humboldt University Berlin and was printed using funds made available by the Deutsche Forschungsgemeinschaft

$(X_0(t),\, t \in J)$ is an initial process with càdlàg trajectories, independent of Z. The question treated in this note concerns the existence of stationary solutions to (1.1).
If $r = 0$, the answer to this question is known. The equation

$$X(t) = X(0) + \rho \int_0^t X(s)\, ds + Z(t), \quad t \geq 0, \tag{1.2}$$

($X(0)$ and Z are independent) admits a stationary solution if and only if

$$\rho < 0 \tag{1.3}$$

and

$$\int_{|y|>1} \log |y|\, F(dy) < \infty, \tag{1.4}$$

where F denotes the Lévy measure of Z. This stationary solution X is called a stationary process of Ornstein–Uhlenbeck type. Its distribution is uniquely determined by ρ and the Lévy-Khintchine characteristics of Z, in particular, the law of $X(t)$ is the distribution of

$$U = \int_0^\infty e^{\rho t} dZ(t).$$

Essentially, these results are due to Wolfe (1982). Their multi-dimensional versions were considered, in particular, by Jurek and Verwaat (1983), Jurek (1982), Sato and Yamazato (1983), Zabczyk (1983), Chojnowska-Michalik (1987).
In this paper we show that a stationary solution of (1.1) exists if and only if the equation

$$h(\lambda) := \lambda - \int_J e^{\lambda u} a(du) = 0 \tag{1.5}$$

has no complex solutions λ with $\operatorname{Re} \lambda \geq 0$, and the condition (1.4) holds. Thus, in comparison with the Ornstein–Uhlenbeck case, the condition (1.3) is replaced by

$$\{\lambda \in \mathbb{C} \mid h(\lambda) = 0,\ \operatorname{Re} \lambda \geq 0\} = \emptyset. \tag{1.6}$$

The distribution of a stationary solution X is unique for given a and the characteristics of Z, and the law of $X(t)$ is the distribution of

$$U = \int_0^\infty x_0(t)\, dZ(t), \tag{1.7}$$

where $x_0(t)$ is the so-called fundamental solution of the corresponding to (1.1) deterministic homogeneous equation (see the definition in Section 2). If Z is a Wiener process and a is concentrated in the points 0 and r, these results were proved by Küchler and Mensch (1992).
As in the case of the equation (1.2), a stationary solution of (1.1) exists if and only if the integral in (1.7) converges in an appropriate sense. But, unlike the Ornstein–Uhlenbeck case (where $x_0(t) = e^{\rho t}$), the fundamental solution $x_0(t)$ is not necessarily

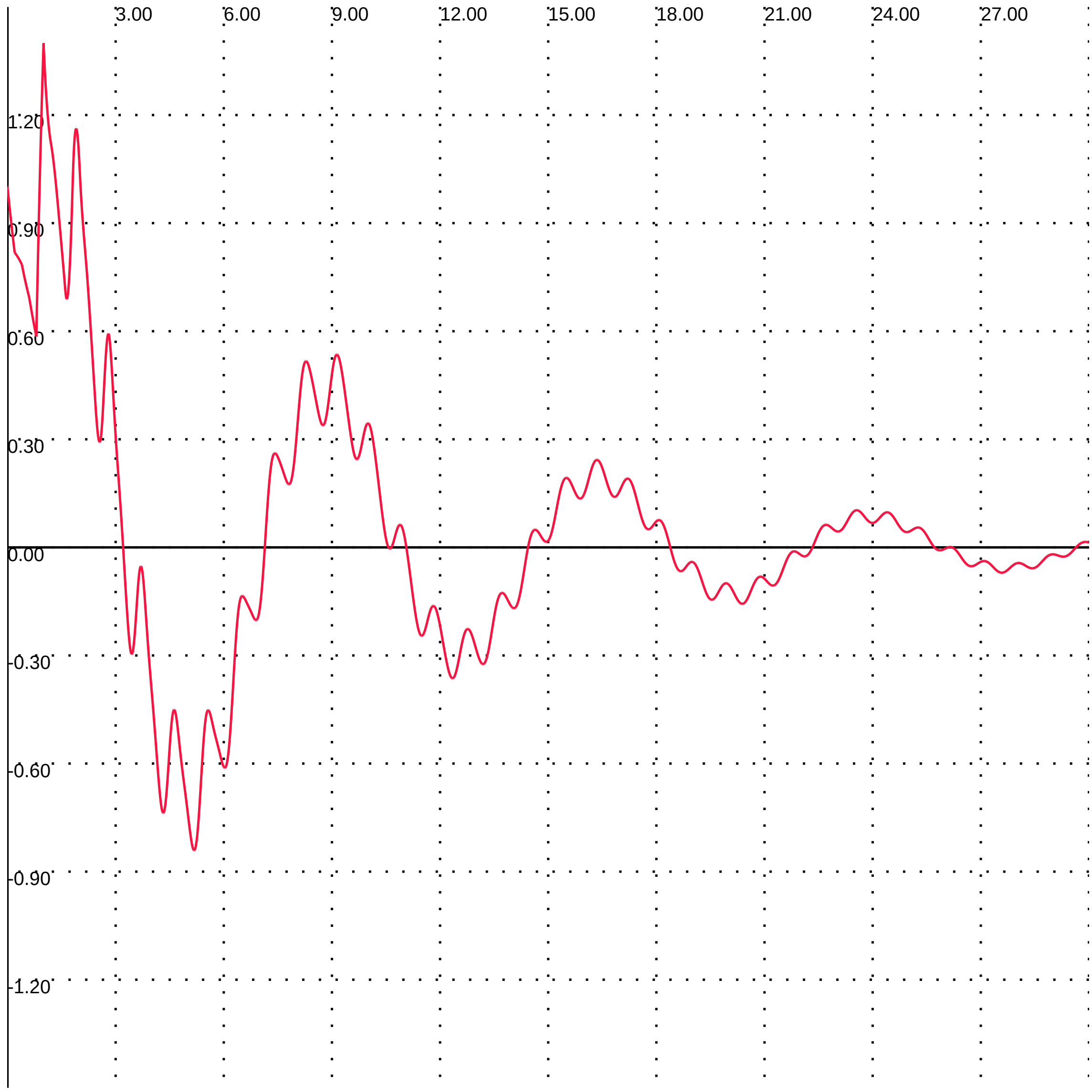

Figure 1: The fundamental solution $x_0(t)$ for $a(du) = -\epsilon_0(du) + 0.7\epsilon_{-0.2}(du) - 0.3\epsilon_{-0.4}(du) - 0.2\epsilon_{-0.6}(du) + 5.5\epsilon_{-0.8}(du) - 5.4\epsilon_{-1}(du)$

a positive monotone function, for example, it may oscillate around 0 under (1.6), see Figure 1. Thus, the proof of the necessity of (1.6) and (1.4) for the convergence of the integral in (1.7) is not so straightforward as in the case $r = 0$.
Stochastic differential equations of the type (1.1) can be considered as linear stochastic differential equations in some Hilbert space $\mathcal{H}$:

$$dX_t = AX_t\,dt + dZ_t, \quad t \geq 0, \tag{1.8}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ of bounded linear operators on $\mathcal{H}$ and $(Z_t)_{t\geq 0}$ is an $\mathcal{H}$-valued Lévy process, see e.g. Da Prato and Zabczyk (1992) for the details. Chojnowska-Michalik (1987) studied the problem of the existence of stationary distributions for the solutions of (1.8) and obtained the sufficiency of conditions similar to (1.6) and (1.4). Under an additional assumption on the semigroup $(T_t)_{t\geq 0}$ ((T_t) can be extended to a group on $\mathbb{R}$), which is not satisfied in our case, she proved also the necessity of these conditions.
The assumption that the initial process X_0 and Z are independent is important for the above result. Otherwise, (1.6) is not necessary for the existence of a stationary solution, cf. Theorem 3.1 in Jacod (1985) and Theorem 20 in Mohammed and Scheutzow (1990).

2 Preliminaries

The aim of this section is twofold: to establish our notation and to recall some basic facts concerning Lévy processes and deterministic delay differential equations of the considered type.

2.1 Deterministic delay differential equations

Since the equation (1.1) involves no stochastic integrals and is treated pathwise, we will formulate a number of results for solutions of the equation (1.1) with deterministic Z and X_0, for which we refer to Hale and Verduyn Lunel (1993), Diekmann *et al.* (1995), and also to Mohammed and Scheutzow (1990).
By a solution of the equation (1.1) we call a real-valued function $X(t)$, $t \geq -r$, which is locally integrable and satisfies (1.1) for all $t \geq -r$ or only for $t \geq 0$ if the initial condition is not specified (here and below "integrable" means "integrable with respect to the Lebesgue measure"; the double integral in (1.1) exists for such functions by the Fubini theorem).
Assume that a finite signed measure a on J, a real-valued locally integrable function Z on $\mathbb{R}_+$ satisfying $Z(0) = 0$, and a real-valued integrable function X_0 on J are given (only such a, Z, and X_0 will be considered in the sequel). Then the equation (1.1) has a unique solution. This solution is càdlàg (resp. continuous, resp. absolutely continuous) if and only if Z is càdlàg (resp. continuous, resp. absolutely continuous).
Given a measure a, we call a function $x_0\colon [-r, \infty[\to \mathbb{R}$ the fundamental solution of the

homogeneous equation

$$\begin{aligned} X(t) &= X(0) + \int_0^t \int_J X(s+u)\, a(du)\, ds, && t \geq 0, \\ X(t) &= X_0(t), && t \in J, \end{aligned} \tag{2.1}$$

if it is the solution of (2.1) corresponding to the initial condition

$$X_0(t) = \begin{cases} 1, & t = 0, \\ 0, & -r \leq t < 0. \end{cases}$$

In other words, a function $x_0(t)$, $t \geq -r$, is the fundamental solution of (2.1) if it is absolutely continuous, $x_0(t) = 0$ for $t < 0$, $x_0(0) = 1$, and

$$\dot{x}_0(t) = \int_J x_0(t+u)\, a(du) \tag{2.2}$$

for Lebesgue-almost all $t > 0$. To facilitate some notation in the sequel it is convenient to put $x_0(t) = 0$ for $t < -r$.
The solution of (1.1) can be represented via the fundamental solution x_0 of (2.1):

$$X(t) = \begin{cases} x_0(t)X_0(0) + \int_J \int_{-u}^0 X_0(s)x_0(t-u-s)\, ds\, a(du) + \int_{[0,t]} Z(t-s)\, dx_0(s), & t \geq 0, \\ X_0(t), & t \in J. \end{cases} \tag{2.3}$$

Remark: The domain of integration in the last integral in (2.3) includes zero:

$$\int_{[0,t]} Z(t-s)\, dx_0(s) = Z(t) + \int_{]0,t]} Z(t-s)\, dx_0(s).$$

The asymptotic behaviour of solutions of the equations (1.1) and (2.1) for $t \to \infty$ is connected with the set of complex solutions of the so-called characteristic equation

$$h(\lambda) = 0, \tag{2.4}$$

where the function $h(\cdot)$ is defined in (1.5). Note that a complex number λ solves (2.4) if and only if $(e^{\lambda t},\ t \geq -r)$ solves (2.1) for the initial condition $X_0(t) = e^{\lambda t}$, $t \in J$.
The set $\Lambda := \{\lambda \in \mathbb{C} \mid h(\lambda) = 0\}$ is not empty; moreover, it is infinite except the case where a is concentrated at 0. Since $h(\cdot)$ is an entire function, Λ consists of isolated points only. It is easy to check that $\lambda_n \in \Lambda$ and $|\lambda_n| \to \infty$ imply $\operatorname{Re} \lambda_n \to -\infty$, thus the set $\{\lambda \in \Lambda \mid \operatorname{Re} \lambda \geq c\}$ is finite for every $c \in \mathbb{R}$. In particular, it holds

$$v_0 := \max\{\, \operatorname{Re} \lambda \mid \lambda \in \Lambda \,\} < \infty. \tag{2.5}$$

Define

$$v_{i+1} := \max\{\, \operatorname{Re} \lambda \mid \lambda \in \Lambda,\ \operatorname{Re} \lambda < v_i \,\}, \qquad i \geq 0.$$

For $\lambda \in \Lambda$ denote by $m(\lambda)$ the multiplicity of λ as a solution of (2.4).
It is easy to check from (2.2) that $1/h(\lambda)$ is the Laplace transform of $(x_0(t),\ t \geq 0)$ at least if $\operatorname{Re}\lambda$ is large enough. (In fact,

$$1/h(\lambda) = \int_0^\infty e^{-\lambda t} x_0(t)\, dt$$

if $\operatorname{Re}\lambda > v_0$.) Applying a standard method based on the inverse Laplace transform and Cauchy's residue theorem, we come to the following lemma which is essentially known and can be found in a slightly different form in Hale and Verduyn Lunel (1993) and Diekmann *et al.* (1995). The proof will be sketched in Section 4.

Lemma 2.1 *For any $c \in \mathbb{R}$ we have*

$$x_0(t) = \sum_{i: v_i \geq c} \left[\sum_{\substack{\lambda \in \Lambda \\ \lambda = v_i}} p_\lambda(t) e^{v_i t} + \sum_{\substack{\lambda \in \Lambda \\ \operatorname{Re}\lambda = v_i \\ \operatorname{Im}\lambda > 0}} \{q_\lambda(t)\cos(t \operatorname{Im}\lambda) + r_\lambda(t)\sin(t\operatorname{Im}\lambda)\}\, e^{v_i t} \right] + o(e^{ct}),$$

$t \to \infty$, where $p_\lambda(t)$ is a real-valued polynomial in t of degree $m(\lambda) - 1$, $q_\lambda(t)$ and $r_\lambda(t)$ are real-valued polynomials in t of degree less than or equal to $m(\lambda) - 1$, and the degree of either $q_\lambda(t)$ or $r_\lambda(t)$ is equal to $m(\lambda) - 1$.

This lemma and the following corollary describe properties of the fundamental solution $x_0(t)$, which are crucial for the proof of our main result.

Corollary 2.2 *For some $\delta > 0$,*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}(|x_0(s)| \geq \delta e^{v_0 s})\, ds > 0.$$

2.2 Lévy processes

Let $Z = (Z(t),\ t \geq 0)$ be a Lévy process. Throughout the paper a continuous truncation function g is fixed, i.e. $g\colon \mathbb{R} \to \mathbb{R}$ is a bounded continuous function with compact support satisfying $g(y) = y$ in a neighbourhood of 0.
It is well known, see e.g. Jacod and Shiryaev (1987), that the distribution of Z is completely characterized by a triple (b, c, F) of the Lévy–Khintchine characteristics, namely, a number $b \in \mathbb{R}$ (the drift), a nonnegative number $c \in \mathbb{R}_+$ (the variance of the Gaussian part), and a nonnegative σ-finite measure F on $\mathbb{R}$ that satisfies $F(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (y^2 \wedge 1) F(dy) < \infty \tag{2.6}$$

(the Lévy measure of jumps). In particular,

$$E \exp\{iu(Z(t) - Z(s))\} = \exp\{(t-s)\psi_{b,c,F}(u)\}, \quad u \in \mathbb{R}, \quad s < t,$$

where

$$\psi_{b,c,F}(u) := iub - \frac{1}{2}u^2c + \int_{\mathbb{R}} (e^{iuy} - 1 - iug(y))F(dy). \tag{2.7}$$

Moreover, this triple (b, c, F) is unique, and, for every triple (b, c, F) satisfying the above assumptions, there is a Lévy process Z with the characteristics (b, c, F).
In the following we shall deal with integrals of the form

$$I_f(t) := \int_0^t f(s)\, dZ(s),$$

where $f: \mathbb{R}_+ \to \mathbb{R}$ is a càdlàg function of locally bounded variation. In this simple case there is no need to use an advanced theory of stochastic integration (however, let us mention that the results stated below are valid for at least locally bounded measurable f). Indeed, the integral $I_f(t)$ can be defined by formal integration by parts:

$$I_f(t) = f(t)Z(t) - \int_{]0,t]} Z(s-)\, df(s), \tag{2.8}$$

where $Z(s-) = \lim_{s' \uparrow s} Z(s')$. Of course, this pathwise definition is equivalent to the usual definitions of stochastic integrals.
The next lemma is a simple exercise. The first equality in its statement can be found e.g. in Lukacs (1969).

Lemma 2.3 *The integral $I_t(f)$ has an infinitely divisible distribution:*

$$E \exp\{iuI_f(t)\} = \exp\left\{ \int_0^t \psi_{b,c,F}(uf(s))\, ds \right\} = \exp\{\psi_{B(t),C(t),F(t)}(u)\},$$

where

$$B(t) := b\int_0^t f(s)\, ds + \int_{\mathbb{R}} \int_0^t \{g(yf(s)) - f(s)g(y)\}\, ds\, F(dy), \tag{2.9}$$

$$C(t) := c\int_0^t f^2(s)\, ds, \tag{2.10}$$

$$F(t; \{0\}) = 0, \qquad \int_{\mathbb{R}} \chi(y)\, F(t; dy) = \int_{\mathbb{R}} \int_0^t \chi(yf(s))\, ds\, F(dy) \tag{2.11}$$

for any nonnegative measurable function χ satisfying $\chi(0) = 0$.

Lemma 2.4 *$I_f(t)$ converges in distribution as $t \to \infty$ if and only if there exist finite limits*

$$B(\infty) := \lim_{t\to\infty} B(t), \qquad C(\infty) := \lim_{t\to\infty} C(t),$$

and

$$\sup_t \int_{\mathbb{R}} (y^2 \wedge 1)\, F(t; dy) < \infty.$$

Moreover, in that case the limit $\lim\limits_{t\to\infty} I_f(t) =: \int\limits_0^\infty f(s)\,dZ(s)$ *exists almost surely and*

$$E\exp\left\{iu\int\limits_0^\infty f(s)\,dZ(s)\right\} = \exp\left\{\lim_{t\to\infty}\int\limits_0^t \psi_{b,c,F}(uf(s))\,ds\right\} = \exp\{\psi_{B(\infty),C(\infty),F(\infty)}(u)\},$$

where $F(\infty)$ *is a* σ*-finite measure on* $\mathbb{R}$ *such that*

$$\int\limits_{\mathbb{R}} \chi(y)\,F(\infty;dy) = \sup_t \int\limits_{\mathbb{R}} \chi(y)\,F(t;dy)$$

for any nonnegative measurable function χ.

Remark: The assumptions of Lemma 2.4 do not imply the integrability of $\psi_{b,c,F}(uf(s))$ on $[0,\infty[$. Of course, if the Lebesgue integral $\int\limits_0^\infty \psi_{b,c,F}(uf(s))\,ds$ exists, then

$$E\exp\left\{iu\int\limits_0^\infty f(s)\,dZ(s)\right\} = \exp\left\{\int\limits_0^\infty \psi_{b,c,F}(uf(s))\,ds\right\}.$$

3 The main result

In this section we assume that there are fixed a finite signed measure a on J and a triple (b,c,F) of the Lévy–Khintchine characteristics such that either $c>0$ or $F\neq 0$. We say that a process $X=(X(t),\ t\geq -r)$ is a solution to the equation (1.1) if there are a Lévy process $Z=(Z(t),\ t\geq 0)$ with the characteristics (b,c,F) and a process $X_0=(X_0(t),\ t\in J)$ with càdlàg trajectories such that (1.1) holds; moreover Z and X_0 are assumed to be independent. In other words, a càdlàg stochastic process $X=(X(t),\ t\geq -r)$ is a solution to (1.1) if

(1) $Z(t) = X(t) - X(0) - \int\limits_0^t\int\limits_J X(s+u)\,a(du)\,ds$, $t\geq 0$, is a Lévy process with the characteristics (b,c,F);

(2) the processes $X=(X(t),\ t\in J)$ and $Z=(Z(t),\ t\geq 0)$ are independent.

We say that a solution $X=(X(t),\ t\geq -r)$ is a stationary solution to (1.1) if

$$(X(t_k),\ k\leq n) \stackrel{d}{=} (X(t+t_k),\ k\leq n) \tag{3.1}$$

for all $t>0$, $n\geq 1$, $t_1,\ldots,t_n\geq -r$.
Recall that $x_0(\cdot)$ is the fundamental solution of the equation (2.1) and v_0 is defined by (2.5).

Theorem 3.1 *There is equivalence between:*

(i) *the equation* (1.1) *admits a stationary solution*;

(ii) *there is a solution X of* (1.1) *such that $X(t)$ has a limit distribution as $t \to \infty$;*

(iii) *for any solution X of* (1.1), *$X(t)$ has a limit distribution as $t \to \infty$;*

(iv) *$v_0 < 0$ and $\int\limits_{|y|>1} \log |y| \, F(dy) < \infty$.*

Moreover, in that case the distribution of $(X(t+t_k),\ k \le n)$, where $n \ge 1$, $0 \le t_1 < t_2 < \cdots < t_n$ are fixed and $X(t)$ is an arbitrary solution of (1.1), *weakly converges as $t \to \infty$ to the distribution of the vector*

$$\left(\int\limits_{t_n - t_k}^{\infty} x_0(s + t_k - t_n) \, dZ(s),\ k \le n \right), \tag{3.2}$$

where $Z = (Z(s),\ s \ge 0)$ is a Lévy process with the characteristics (b, c, F).

Remarks: 1. The integrals in (3.2) are defined in Lemma 2.4. The correctness of their definition will be shown in Lemma 4.3.
2. It follows from the proof of Theorem 3.1 that, given a Lévy process Z with the characteristics (b, c, F) on a probability space $(\Omega, \mathcal{F}, P)$, one can construct, under the condition (iv), a stationary solution on the same probability space if it is large enough, in particular, if there is another Lévy process on $(\Omega, \mathcal{F}, P)$ with the same characteristics independent of Z.

4 Proofs

Proof of Lemma 2.1: According to Lemma I.5.1 and Theorem I.5.4 in Diekmann *et al.* (1995),

$$x_0(t) = \sum_{\substack{\lambda \in \Lambda \\ \operatorname{Re} \lambda \ge c}} \operatorname*{Res}_{z=\lambda} \frac{e^{zt}}{h(z)} + o(e^{ct}), \quad t \to \infty. \tag{4.1}$$

Let $\lambda \in \Lambda$, $\operatorname{Re} \lambda \ge c$, and $m := m(\lambda)$. Write Laurent's series of $1/h(z)$ at $z = \lambda$ in the form

$$1/h(z) = \sum_{k=-m}^{\infty} A_k(\lambda)(z - \lambda)^k, \quad A_{-m}(\lambda) \ne 0.$$

Since

$$e^{zt} = e^{\lambda t} \sum_{k=0}^{\infty} \frac{t^k}{k!} (z - \lambda)^k,$$

the multiplication of the above series yields

$$\operatorname*{Res}_{z=\lambda} \frac{e^{zt}}{h(z)} = e^{\lambda t} \sum_{k=-m}^{-1} \frac{A_k(\lambda)}{(-1-k)!} t^{-1-k}.$$

Note that $h(\overline{z}) = \overline{h(z)}$ (where a bar means the complex conjugate). Therefore, we have $\lambda \in \Lambda$ if and only if $\overline{\lambda} \in \Lambda$. Moreover, it holds $\overline{A_k(\lambda)} = A_k(\overline{\lambda})$. Hence, if $\operatorname{Im} \lambda = 0$, then $A_k(\lambda) \in \mathbb{R}$ and $p_\lambda(t) = \sum\limits_{k=-m}^{-1} \frac{A_k(\lambda)}{(-1-k)!} t^{-1-k}$. If $\operatorname{Im} \lambda \neq 0$, we join two terms in (4.1) corresponding to λ and $\overline{\lambda}$. After simple calculations we obtain (for defineteness, we assume that $\operatorname{Im} \lambda > 0$)

$$\operatorname*{Res}_{z=\lambda} \frac{e^{zt}}{h(z)} + \operatorname*{Res}_{z=\overline{\lambda}} \frac{e^{zt}}{h(z)} = \{q_\lambda(t) \cos(t \operatorname{Im} \lambda) + r_\lambda(t) \sin(t \operatorname{Im} \lambda)\} e^{t \operatorname{Re} \lambda},$$

where

$$q_\lambda(t) = 2 \sum_{k=-m}^{-1} \frac{\operatorname{Re} A_k(\lambda)}{(-1-k)!} t^{-1-k}, \qquad r_\lambda(t) = -2 \sum_{k=-m}^{-1} \frac{\operatorname{Im} A_k(\lambda)}{(-1-k)!} t^{-1-k}.$$

Proof of Corollary 2.2: According to Lemma 2.1, it is enough to check that, for some $\delta > 0$,

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}(|f(s)| \geq \delta)\, ds > 0,$$

for a continuous function $f(t)$ satisfying

$$f(t) = p(t) + \sum_{j=1}^{n} \{q_j(t) \cos(\xi_j t) + r_j(t) \sin(\xi_j t)\} + o(1), \quad t \to \infty,$$

where $p(t)$, $q_i(t)$, $r_i(t)$, $i = 1, \ldots, n$, are polynomials, not all of them being equal to zero identically, $0 < \xi_1 < \cdots < \xi_n$. Thus,

$$f(t) = t^m \hat{f}(t) + o(t^m), \quad t \to \infty,$$

for some $m \geq 0$ and

$$\hat{f}(t) = A_0 + \sum_{j=1}^{n} \{A_j \cos(\xi_j t) + B_j \sin(\xi_j t)\}, \qquad \text{with} \quad M := |A_0| + \sum_{j=1}^{n} (|A_j| + |B_j|) > 0.$$

Then

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}(|f(s)| \geq \delta)\, ds \quad &\geq \quad \liminf_{t\to\infty} \frac{1}{t} \int_1^t \mathbf{1}(|f(s)| \geq \delta s^m)\, ds \\ &\geq \quad \liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}(|\hat{f}(s)| \geq \hat{\delta})\, ds \end{aligned}$$

for any $\hat{\delta} > \delta$. Since

$$\int_0^t \mathbf{1}(|\hat{f}(s)| \geq \hat{\delta})\, ds \geq \frac{1}{M^2} \int_0^t \hat{f}^2(s) \mathbf{1}(|\hat{f}(s)| \geq \hat{\delta})\, ds \geq \frac{1}{M^2} \int_0^t \hat{f}^2(s)\, ds - \frac{\hat{\delta}^2}{M^2} t,$$

we obtain

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}(|\hat f(s)| \ge \hat\delta)\, ds \;\ge\; \frac{1}{M^2}\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t \hat f^2(s)\,ds - \hat\delta^2\right)$$
$$= \frac{1}{M^2}\left\{A_0^2 + \frac{1}{2}\sum_{j=1}^n (A_j^2 + B_j^2) - \hat\delta^2\right\} > 0$$

for $\hat\delta$ small enough.

Proof of Lemma 2.4: According to well-known conditions for the weak convergence of infinitely divisible distributions (see e.g. Remark VII.2.10 in Jacod and Shiryaev (1987)), $I_f(t)$ converges in distribution as $t\to\infty$ if and only if there is a finite limit $\lim_{t\to\infty} B(t)$ and the measures $C(t)\epsilon_0(dy)+(y^2\wedge 1)F(t;dy)$ weakly converge to a measure $\tilde C\epsilon_0(dy)+(y^2\wedge 1)\tilde F(dy)$ with $\tilde F(\{0\})=0$, the limit distribution being infinitely divisible with the characteristics $(B(\infty),\tilde C,\tilde F)$ (here $\epsilon_0(\cdot)$ is the Dirac measure at 0). In our case $F(t)-F(s)$ is a nonnegative measure for all $t>s$ due to (2.11). Therefore, the conditions just mentioned take place if and only if the conditions of Lemma are satisfied, moreover, $\tilde C = C(\infty)$ and $\tilde F = F(\infty)$. It remains to note that $I_f(t)$ is a càdlàg process with independent increments, hence the convergence in distribution of $I_f(t)$ as $t\to\infty$ implies the convergence of $I_f(t)$ almost surely as $t\to\infty$.

Before proving Theorem 3.1 we need a number of preliminary lemmas. We keep the notation and the conventions of Section 2.

Lemma 4.1 *Assume that $v_0<0$ and $X(t)$ is a solution of the (deterministic) equation (2.1). Then $\lim_{t\to\infty} X(t) = 0$.*

Proof: According to (2.3),

$$X(t) = x_0(t)X_0(0) + \int_J \int_{-u}^0 X_0(s)x_0(t-u-s)\,ds\,a(du), \quad t\ge 0.$$

By Lemma 2.1, $|x_0(t)|\le ce^{-\gamma t}$, $t\ge 0$, for some $c>0$ and γ such that $0<\gamma<|v_0|$, from which the claim follows easily.

Lemma 4.2 *Let $z\colon[0,T]\to\mathbb{R}$, $T\ge 0$, be a càdlàg function. Put*

$$X(t) = x_0(t+T)z(T) - \int_{]0,T]} z(s-)\,dx_0(t+s), \quad t\ge -r. \tag{4.2}$$

Then $(X(t),\ t\ge -r)$ is a càdlàg solution of the homogeneous equation (2.1).

Remark: If z has a bounded variation, integration by parts gives

$$X(t) = \int_0^T x_0(t+s)\,dz(s), \quad t\ge -r,$$

i.e. $X(\cdot)$ is a mixture of $x_0(\cdot+s)$, $s\in[0,T]$. Thus, the statement of the lemma is not surprising since every $x_0(\cdot+s)$ is a solution of (2.1).

Proof: If z is a piecewise constant function, the claim follows immediately from the previous remark. For the general case, use a uniform approximation of z by piecewise constant functions.

Lemma 4.3 *Let $f\colon\mathbb{R}_+\to\mathbb{R}$ be a function of locally bounded variation such that $|f(t)|\le ce^{-\gamma t}$ for some $c>0$ and $\gamma>0$. If (1.4) holds, then $I_f(t)$ has a limit distribution as $t\to\infty$.*

Proof: We will check the conditions of Lemma 2.4. First, in view of (2.10),

$$\lim_{t\to\infty} C(t)=\int_0^\infty f^2(s)\,ds<\infty. \tag{4.3}$$

Let us show that

$$\sup_t \int_{\mathbb{R}} (y^2\wedge 1)\,F(t;dy)<\infty. \tag{4.4}$$

Indeed, by (2.11),

$$\begin{aligned}
\int_{\mathbb{R}} (y^2\wedge 1)\,F(t;dy) &= \int_{\mathbb{R}}\int_0^t (y^2 f^2(s)\wedge 1)\,ds\,F(dy) \le \int_{\mathbb{R}}\int_0^\infty (c^2y^2e^{-2\gamma s}\wedge 1)\,ds\,F(dy)\\
&= c^2\int_{|y|\le c^{-1}}\int_0^\infty y^2e^{-2\gamma s}\,ds\,F(dy)+c^2\int_{|y|>c^{-1}}\int_{\gamma^{-1}\log(c|y|)}^\infty y^2e^{-2\gamma s}\,ds\,F(dy)\\
&\quad+\int_{|y|>c^{-1}}\int_0^{\gamma^{-1}\log(c|y|)} ds\,F(dy)\\
&= (2\gamma)^{-1}c^2\int_{|y|\le c^{-1}} y^2\,F(dy)+\gamma^{-1}\int_{|y|>c^{-1}}(\log c+\log|y|+1/2)\,F(dy).
\end{aligned}$$

The right-hand side of the previous inequality is finite in view of (2.6) and (1.4). In view of (2.9), in order to show that

$$B(t)\to b\int_0^\infty f(s)\,ds+\int_{\mathbb{R}}\int_0^\infty\{g(yf(s))-f(s)g(y)\}\,ds\,F(dy),\quad t\to\infty, \tag{4.5}$$

it is enough to check that

$$\int_{\mathbb{R}}\int_0^\infty |g(yf(s))-f(s)g(y)|\,ds\,F(dy)<\infty. \tag{4.6}$$

Choose a $\kappa>0$ such that $g(y)=y$ if $|y|\le\kappa$. Without loss of generality assume that $c\ge 1$.

Since $|f(s)| \le c$,

$$\int_0^\infty |g(yf(s)) - f(s)g(y)|\, ds = 0 \qquad \text{if} \quad |y| \le \kappa c^{-1}. \tag{4.7}$$

Let $|y| > \kappa c^{-1}$ and put $L = \max\{\sup_{y\in\mathbb{R}} |g(y)|,\ \kappa\}$. Then

$$\begin{aligned}
&\int_0^\infty |g(yf(s)) - f(s)g(y)|\, ds \\
&\qquad \le L \int_0^\infty |f(s)|\, ds + \int_0^\infty (|yf(s)|\mathbf{1}(|yf(s)| \le \kappa) + L\mathbf{1}(|yf(s)| > \kappa))\, ds \\
&\qquad \le \gamma^{-1}Lc + \int_0^\infty (c|y|e^{-\gamma s}\mathbf{1}(s \ge \gamma^{-1}\log(c\kappa^{-1}|y|)) + L\mathbf{1}(s < \gamma^{-1}\log(c\kappa^{-1}|y|)) \\
&\qquad = \gamma^{-1}(Lc + \kappa + L\log(c\kappa^{-1}) + \log|y|).
\end{aligned} \tag{4.8}$$

Now (4.6) follows from (4.7), (4.8), (2.6), and (1.4), and the statement follows from (4.3)–(4.5) and Lemma 2.4.

Lemma 4.4 *Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a locally bounded measurable function such that*

$$\int_{\mathbb{R}} \int_0^\infty (y^2 f^2(s) \wedge 1)\, ds\, F(dy) < \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}(|f(s)| \ge \delta e^{-\gamma s})\, ds > 0$$

for some $\delta > 0$ and $\gamma > 0$. Then (1.4) holds.

Proof: Put

$$G(t) = \int_0^t \mathbf{1}(|f(s)| \ge \delta e^{-\gamma s})\, ds.$$

By the assumption, there are a $T > 0$ and an $\varepsilon > 0$ such that $G(t) \ge \varepsilon t$ for all $t \ge T$. We have

$$\begin{aligned}
\int_{\mathbb{R}} \int_0^\infty (y^2 f^2(s) \wedge 1)\, ds\, F(dy) &\ge \int_{|y|\ge\delta^{-1}e^{\gamma T}} \int_0^{\gamma^{-1}\log(\delta|y|)} (y^2 f^2(s) \wedge 1)\mathbf{1}(|f(s)| \ge \delta e^{-\gamma s})\, ds\, F(dy) \\
&= \int_{|y|\ge\delta^{-1}e^{\gamma T}} G(\gamma^{-1}\log(\delta|y|))\, F(dy) \\
&\ge \varepsilon\gamma^{-1} \int_{|y|\ge\delta^{-1}e^{\gamma T}} \log(\delta|y|)\, F(dy).
\end{aligned}$$

The left-hand side of the above inequality is finite by the assumptions, so we easily obtain (1.4).

Proof of Theorem 3.1: Let us first note that by (2.8),

$$\int_{[0,t]} Z(t-s)\,dx_0(s) = \int_0^t x_0(t-s)\,dZ(s).$$

Thus, using (2.3), any solution of the equation (1.1) can be written in the form

$$X(t) = x_0(t)X_0(0) + \int_J \int_{-u}^{0} X_0(s)x_0(t-u-s)\,ds\,a(du) + \int_0^t x_0(t-s)\,dZ(s), \quad t \geq 0. \tag{4.9}$$

Note also that, by Lemma 2.3,

$$\int_0^t x_0(t-s)\,dZ(s) \stackrel{d}{=} \int_0^t x_0(s)\,dZ(s). \tag{4.10}$$

The implications (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(ii) are trivial.
Let us prove (iv)$\Rightarrow$(iii). According to Lemma 4.1, the first two summands in the right-hand side of (4.9) converge to zero for all ω. Now (iii) follows from (4.10), Lemmas 4.3 and 2.1. The same argument shows that the limit distribution of $(X(t+t_k),\ k \leq n)$, $0 \leq t_1 < \cdots < t_n$, $t \to \infty$, coincides with the limit distribution of $(\int_0^{t+t_k} x_0(t+t_k-s)\,dZ(s),$ $k \leq n)$. By Lemma 2.3,

$$\Big(\int_0^{t+t_k} x_0(t+t_k-s)\,dZ(s),\ k \leq n\Big) \stackrel{d}{=} \Big(\int_{t_n-t_k}^{t+t_n} x_0(s+t_k-t_n)\,dZ(s),\ k \leq n\Big),$$

and the vector on the right has the limit distribution (3.2) by Lemmas 2.4 and 4.3.
Our next step is to prove (iv)$\Rightarrow$(i). Let $Z = (Z(t),\ t \geq 0)$ and $\tilde{Z} = (\tilde{Z}(t),\ t \geq 0)$ be two independent Lévy processes with the same characteristics (b, c, F). To make the idea more clear, let us define a two-sided Lévy process $(Z(t),\ t \in \mathbb{R})$ by

$$Z(t) = \begin{cases} Z(t), & t \geq 0, \\ \tilde{Z}(-t-0), & t < 0, \end{cases}$$

and put

$$X(t) = \int_{-\infty}^{t} x_0(t-s)\,dZ(s) := \begin{cases} \int_0^t x_0(t-s)\,dZ(s) + \int_0^\infty x_0(t+s)\,d\tilde{Z}(s), & t \geq 0, \\ \int_0^\infty x_0(t+s)\,d\tilde{Z}(s), & -r \leq t < 0. \end{cases} \tag{4.11}$$

The process $X = (X(t),\ t \geq -r)$ is well defined up to a modification according to Lemmas 2.4, 4.3 and 2.1. Moreover, let $-r \leq t_1 < \cdots < t_n$. By Lemmas 2.3 and 2.4,

$$\begin{aligned}
E\exp\left(i\sum_{j=1}^{n} u_j X(t_j)\right) &= \exp\Big(\int_0^\infty \psi_{b,c,F}\Big(\sum_{j=1}^{n} u_j x_0(t_j+s)\Big)ds \\
&\qquad + \mathbf{1}(t_n>0)\int_0^{t_n} \psi_{b,c,F}\Big(\sum_{j:t_j>0} u_j x_0(t_j-s)\Big)ds\Big) \\
&= \exp\Big(\int_{t_n}^\infty \psi_{b,c,F}\Big(\sum_{j=1}^{n} u_j x_0(s-t_n+t_j)\Big)ds \\
&\qquad + \mathbf{1}(t_n>0)\int_0^{t_n} \psi_{b,c,F}\Big(\sum_{j:t_j>0} u_j x_0(s-t_n+t_j)\Big)ds\Big) \\
&= \exp\Big(\int_0^\infty \psi_{b,c,F}\Big(\sum_{j=1}^{n} u_j x_0(s-t_n+t_j)\Big)ds\Big).
\end{aligned}$$

Therefore, the process X is stationary in the sense of (3.1), and we need to prove that it has a càdlàg modification satisfying (1.1).
Formally, we proceed as follows. For an integer $N \geq r$ we define

$$X_N(t) = \begin{cases} \int\limits_{[0,t]} Z(t-s)\,dx_0(s) + x_0(N+t)\widetilde{Z}(N) - \int\limits_{]0,N]} \widetilde{Z}(s-)\,dx_0(t+s), & t \geq 0, \\ x_0(N+t)\widetilde{Z}(N) - \int\limits_{]0,N]} \widetilde{Z}(s-)\,dx_0(t+s), & -r \leq t < 0. \end{cases} \tag{4.12}$$

Combining (2.3) and Lemma 4.2, we obtain that $(X_N(t),\ t \geq -r)$ is a càdlàg solution to the equation (1.1). By (2.8), (4.12) can be rewritten in the form

$$X_N(t) = \begin{cases} \int_0^t x_0(t-s)\,dZ(s) + \int_0^N x_0(t+s)\,d\widetilde{Z}(s), & t \geq 0, \\ \int_0^N x_0(t+s)\,d\widetilde{Z}(s), & -r \leq t < 0. \end{cases} \tag{4.13}$$

Comparing the last equality with (4.11), we conclude that

$$\lim_{N\to\infty} X_N(t) = X(t)$$

with probability one for every fixed $t \geq -r$ by Lemmas 2.4, 4.3 and 2.1. Hence, to prove (i) it is sufficient to check that the series

$$\sum_N \{X_{N+1}(t) - X_N(t)\}$$

converges uniformly in t for almost all ω.

It follows from (4.12) and (4.13) that

$$
\begin{aligned}
X_{N+1}(t)-X_N(t) &= x_0(N+1+t)(\widetilde{Z}(N+1)-\widetilde{Z}(N)) \\
&\qquad -\int_N^{N+1}(\widetilde{Z}(s-)-\widetilde{Z}(N))\,dx_0(t+s) \\
&= \int_N^{N+1} x_0(t+s)\,d\widetilde{Z}(s).
\end{aligned}
\tag{4.14}
$$

Since $v_0<0$ in our case, by Lemma 2.1 and (2.2),

$$
|x(t)|\le ce^{-\gamma t},\quad t\ge -r,\qquad |\dot{x}(t)|\le ce^{-\gamma t},\quad t\ge 0, \tag{4.15}
$$

for some $\gamma\in]0,-v_0[$ and $c>0$.
It is well known that the Lévy process $\widetilde{Z}$ can be decomposed into the sum

$$
\widetilde{Z}=bt+M(t)+\sum_{0<s\le t}\Delta\widetilde{Z}(s)\mathbf{1}(|\Delta\widetilde{Z}(s)|>1),
$$

where $\Delta\widetilde{Z}(s)=\widetilde{Z}(s)-\widetilde{Z}(s-)$ and $M(t)$ is a square-integrable martingale with the quadratic characteristic $\Big(c+\int_{|y|\le 1} y^2\,F(dy)\Big)t$, see e.g. Jacod and Shiryaev (1987), Chapter II. Thus, (4.14) and (4.15) yield

$$
\begin{aligned}
|X_{N+1}(t)-X_N(t)| &\le \Big(|x_0(N+1+t)|+\int_N^{N+1}|\dot{x}_0(t+s)|\,ds\Big)(|b|+\zeta_N) \\
&\qquad +\sum_{N<s\le N+1}|x_0(t+s)|\Delta\widetilde{Z}(s)\mathbf{1}(\Delta\widetilde{Z}(s)>1) \\
&\qquad -\sum_{N<s\le N+1}|x_0(t+s)|\Delta\widetilde{Z}(s)\mathbf{1}(\Delta\widetilde{Z}(s)<-1) \\
&\le 2ce^{\gamma r}e^{-\gamma N}(|b|+\zeta_N) \\
&\qquad +ce^{\gamma r}\sum_{N<s\le N+1}e^{-\gamma s}\Big\{\Delta\widetilde{Z}(s)\mathbf{1}(\Delta\widetilde{Z}(s)>1) \\
&\qquad\qquad -\Delta\widetilde{Z}(s)\mathbf{1}(\Delta\widetilde{Z}(s)<-1)\Big\},
\end{aligned}
$$

where $\zeta_N=\sup_{s\in[N,N+1]}|M(s)-M(N)|$. By Doob's inequality,

$$
E\zeta_N^2\le 4E(M(N+1)-M(N))^2=4\Big(c+\int_{|y|\le 1}y^2\,F(dy)\Big)<\infty. \tag{4.16}
$$

Thus, the series $\sum_N e^{-\gamma N}(|b|+\zeta_N)$ converges almost surely since $\sum_N e^{-\gamma N}(|b|+E\zeta_N)<\infty$ in view of (4.16). Finally, the series

$$
\sum_{s>0}e^{-\gamma s}\Big\{\Delta\widetilde{Z}(s)\mathbf{1}(\Delta\widetilde{Z}(s)>1)-\Delta\widetilde{Z}(s)\mathbf{1}(\Delta\widetilde{Z}(s)<-1)\Big\}
$$

is also converging almost surely by Lemma 4.3, since

$$\sum_{s>0} \Delta\widetilde{Z}(s)\mathbf{1}(\Delta\widetilde{Z}(s) > 1) \qquad \text{and} \qquad \sum_{s>0} \Delta\widetilde{Z}(s)\mathbf{1}(\Delta\widetilde{Z}(s) < -1)$$

are Lévy processes with the Lévy measures $\mathbf{1}(y > 1)\, F(dy)$ and $\mathbf{1}(y < -1)\, F(dy)$ respectively.

Our last step is to prove (ii)⇒(iv). Let X be a solution of (1.1) such that $X(t)$ converges in distribution as $t \to \infty$. Let $\varphi_t(u)$, $u \in \mathbb{R}$, be the characteristic function of $X(t)$. Then there is an interval $[0, u_0]$, $u_0 > 0$ and numbers $\delta \in]0, 1[$ and $t_0 \geq 0$ such that $|\varphi_t(u)| \geq \delta$ for all $u \in [0, u_0]$ and $t \geq t_0$.

In view of (4.9), (4.10) and independence of X_0 and Z,

$$\left| E \exp\left(iu \int_0^t x_0(s)\, dZ(s) \right) \right| \geq |\varphi_t(u)| \geq \delta, \quad u \in [0, u_0], \quad t \geq t_0. \tag{4.17}$$

Let $(B(t), C(t), F(t))$ be the Lévy–Khintchine characteristics of the distribution of $\int_0^t x_0(s)\, dZ(s)$, i.e.

$$E \exp\left(iu \int_0^t x_0(s)\, dZ(s) \right) = \exp\left(iuB(t) - \frac{1}{2}u^2 C(t) + \int_{\mathbb{R}} (e^{iuy} - 1 - iug(y))\, F(t; dy) \right). \tag{4.18}$$

We obtain from (4.17) and (4.18) that

$$\frac{u^2}{2} C(t) + \int_{\mathbb{R}} (1 - \cos(uy))\, F(t; dy) \leq L := -\log \delta, \quad u \in [0, u_0]. \tag{4.19}$$

Let $F = 0$. Then $c > 0$ by our assumptions and $C(t) = c \int_0^t x_0^2(s)\, ds$ by (2.10). Hence, $\int_0^\infty x_0^2(s)\, ds < \infty$ by (4.19) and $v_0 < 0$ by Corollary 2.2.

Let $F \neq 0$. Integrating (4.19) over u from 0 to u_0, we get

$$\int_{\mathbb{R}} \left(u_0 - \frac{\sin(u_0 y)}{y} \right) F(t; dy) \leq L u_0.$$

Taking into account that

$$y^2 \wedge 1 \leq \kappa \left(u_0 - \frac{\sin(u_0 y)}{y} \right)$$

for all $y \neq 0$, where κ is a positive constant (depending on u_0), and using (2.11) and (4.19), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_0^\infty (y^2 x_0^2(s) \wedge 1)\, ds\, F(dy) &= \lim_t \int_{\mathbb{R}} \int_0^t (y^2 x_0^2(s) \wedge 1)\, ds\, F(dy) \\ &= \lim_t \int_{\mathbb{R}} (y^2 \wedge 1)\, ds\, F(t; dy) \leq \kappa L u_0 < \infty. \end{aligned}$$

By Corollary 2.2, if $v_0 \geq 0$ then $\int\limits_0^\infty (y^2 x_0^2(s) \wedge 1)\, ds = \infty$ for all $y \neq 0$. Thus, $v_0 < 0$ and because of Corollary 2.2 the function $x_0(t)$ satisfies the assumptions of Lemma 4.4, which yields (iv).

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